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RADIUS OF STRONGLY STARLIKENESS FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. We determine the radius of p-valent strongly starlike of order γ for certain polynomials of the form $F(z)=f(z)\cdot[Q(z)]^{\frac{\beta}{n}}$.

1. Introduction

Let A_p (p is fixed integer ≥ 1) denote the class of functions $f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k$ which are analytic in the unit disk $D=\{z\in D:|z|<1\}$. Let Ω denote the class of bounded function $w(z)$ analytic in D and satisfying the conditions $w(0)=0$ and $|w(z)|\leq|z|$, $z\in D$. We use P to denote the class of functions $p(z)=1+c_1z+c_2z^2+\cdots$ which are analytic in D and a positive real part there.

For $0\leq\alpha<p$ and $|\lambda|<\frac{\pi}{2}$, we denote by $S_p^{\lambda}(\alpha)$, the family of functions $g(z)\in A_p$ which satisfy

$$\frac{zg'(z)}{g(z)}\prec\frac{p+\{2(p-\alpha)\cos\lambda\cdot\exp(-i\lambda)-p\}z}{1-z},\quad z\in D \tag{1.1}$$

where $\prec$ means subordination. From the definition of subordination it follows that $g(z)\in A_p$ has a representation

$$\frac{zg'(z)}{g(z)}=\frac{p+\{2(p-\alpha)\cos\lambda\cdot\exp(-i\lambda)-p\}w(z)}{1-w(z)}$$

where $w(z)\in\Omega$. Clearly, $S_p^{\lambda}(\alpha)$ is subclass of p-valent λ-spiral functions of order α. For $\lambda=0$, we have the class $S_p^*(\alpha)$, $0\leq\alpha<p$, of p-valent starlike functions of order α, investigated by Goluzina [3].

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As noted in a function is p-valent strongly starlike of order γ, $0 < \gamma \leq 1$ if

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| \leq \frac{\pi}{2}\gamma.$$

Basgőze(1969) has obtained sharp inequalities of univalence(starlikeness) for certain polynomials of the form $F(z) = f(z) \cdot [Q(z)]^{\frac{\beta}{n}}$, where β is real and $Q(z)$ is a polynomial of degree $n > 0$ all of whose zeros are outside or on the unit circle $\{z \in D : |z| = 1\}$. Rajasekaran [5] extended Basgőze's results for certain classes of analytic functions of the form. Recently, J. Patel [4] generalized some of the work of Rajasekaran and Basgőze for functions belonging to the class $S_p^{\lambda}(\alpha)$. That is, determine the radius of starlikeness for some classes of p-valent analytic functions of the polynomial form $F(z)$.

In the present paper, we will extend the results of J. patel. Thus, we determine the radius of p-valent strongly starlike of order γ for the polynomials of the form $F(z)$ in the such problems.

2. Some Lemmas

Before proving our next results, we need the following Lemmas.

Lemma 1 (A. Gangadharan [2]). *For $|z| \leq r < 1$, $|z_k| = R > r$, we have*

$$\left|\frac{z}{z - z_k} + \frac{r^2}{R^2 - r^2}\right| \leq \frac{Rr}{R^2 - r^2}.$$

Lemma 2 (Ratti [6]). *If $\phi(z)$ is analytic in D and $|\phi(z)| \leq 1$ for $z \in D$, then for $|z| = r < 1$,*

$$\left|\frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)}\right| \leq \frac{1}{1 - r}.$$

Lemma 3 (Causey and Merke's [1]). *If $p(z) = 1 + c_1 z + c_2 z + \cdots \in P$, then for $|z| = r < 1$,*

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2r}{1 - r^2}.$$

This estimate is sharp.

Lemma 4 (J. Patel [4]). *Suppose $g(z) \in S_p^\lambda(\alpha)$. Then for $|z| = r < 1$,*

$$\left| \frac{zg'(z)}{g(z)} - \left\{ p + \frac{2(p-\alpha)e^{i\lambda}r^2 \cos\lambda}{1-r^2} \right\} \right| \leq \frac{2(p-\alpha)r\cos\lambda}{1-r^2}.$$

The result is sharp.

Lemma 5 (A. Gangadharan [2]). *If $R_a \leq (Re\ a)\sin\left(\frac{\pi}{2}\gamma\right) - (Im\ a)\cos\left(\frac{\pi}{2}\gamma\right)$, $Im\ a \geq 0$, the disk $|w - a| \leq Ra$ is contained in the sector $|\arg w| \leq \frac{\pi}{2}\gamma$, $0 < \gamma \leq 1$.*

3. Main Theorem

Theorem 1. *Suppose*

$$F(z) = f(z)[Q(z)]^{\frac{\beta}{n}} \tag{3.1}$$

where β is real and $Q(z)$ is a polynomial of degree $n > 0$ with no zeros in $|z| < R$, $R \geq 1$. If $f(z) \in A_p$ satisfies

$$Re\left[\left(\frac{f(z)}{g(z)}\right)^{\frac{1}{\delta}}\right] > 0, \quad 0 < \delta \leq 1, \quad z \in D \tag{3.2}$$

and

$$Re\left[\frac{g(z)}{h(z)}\right] > 0, \quad z \in D \tag{3.3}$$

for some $g(z) \in A_p$ and $h(z) \in S_p^\lambda(\alpha)$, then $F(z)$ is p-valent strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

$$\begin{aligned}
&r^4\left[(p+\beta)\sin\frac{\pi}{2}\gamma + 2(p-\alpha)\cos\lambda\sin(\lambda - \frac{\pi}{2}\gamma)\right] \\
&+r^3[|\beta|R + 2(p-\alpha)\cos\lambda + 2(\delta+1)] \\
&-r^2\left[(p(1+R^2)+\beta)\sin\frac{\pi}{2}\gamma + 2(p-\alpha)R^2\cos\lambda\sin(\lambda - \frac{\pi}{2}\gamma)\right] \\
&-r[|\beta|R + 2(p-\alpha)R^2\cos\lambda + 2(\delta+1)R^2] \\
&+pR^2\sin\frac{\pi}{2}\gamma.
\end{aligned} \tag{3.4}$$

Proof. We choose a suitable branch of $(f(z)/g(z))^{\frac{1}{\delta}}$ so that $(f(z)/g(z))^{\frac{1}{\delta}}$ is analytic in D and takes the value 1 at $z=0$. Thus form (3.2) and (3.3), we have

$$F(z) = p_1^{\delta}(z)p_2h(z)[Q(z)]^{\frac{\beta}{n}} \tag{3.5}$$

where $p_j(z) \in P$ $(j=1,2)$.

Then we have

$$\frac{zF'(z)}{F(z)} = \delta\frac{zp_1'(z)}{p_1(z)} + \frac{zp_2'(z)}{p_2(z)} + \frac{zh'(z)}{h(z)} + \frac{\beta}{n}\sum_{k=1}^{n}\frac{z}{z-z_k}. \tag{3.6}$$

Since $h(z) \in S_p^{\lambda}(\alpha)$, by Lemma 4, we have

$$\left|\frac{zh'(z)}{h(z)} - \left\{p + \frac{2(p-\alpha)e^{i\lambda}r^2\cos\lambda}{1-r^2}\right\}\right| \le \frac{2(p-\alpha)r\cos\lambda}{1-r^2}. \tag{3.7}$$

Using (3.6) and (3.7) an Lemma 1, 3, we get

$$\begin{aligned}&\left|\frac{zF'(z)}{F(z)} - \left\{p + \frac{2(p-\alpha)e^{i\lambda}r^2\cos\lambda}{1-r^2} - \frac{\beta r^2}{R^2-r^2}\right\}\right| \\ \le & \frac{2\{(p-\alpha)r\cos\lambda + r(\delta+1)\}}{1-r^2} + \frac{|\beta|Rr}{R^2-r^2}.\end{aligned} \tag{3.8}$$

Using Lemma 5, we get that the about disk is contained in the sector $|\arg w| < \frac{\pi}{2}\gamma$ provided the inequality

$$\begin{aligned}&\frac{2\{(p-\alpha)r\cos\lambda + r(\delta+1)\}}{1-r^2} + \frac{|\beta|Rr}{R^2-r^2} \\ &\le \left\{p + \frac{2(p-\alpha)r^2\cos^2\lambda}{1-r^2} - \frac{\beta r^2}{R^2-r^2}\right\}\sin\frac{\pi}{2}\gamma - \frac{2(p-\alpha)r^2\sin\lambda\cos\lambda}{1-r^2}\cos\frac{\pi}{2}\gamma\end{aligned}$$

is satisfied. The above inequality simplifies to $T(r) \ge 0$, where

$$\begin{aligned}T(r) =& r^4\left[(p - 2(p-\alpha)\cos^2\lambda + \beta)\sin\frac{\pi}{2}\gamma + (p-\alpha)\sin 2\lambda\cos\frac{\pi}{2}\gamma\right] \\ &+ r^3[|\beta|R + 2(p-\alpha)\cos\lambda + 2(\delta+1)] \\ &+ r^2\left[(-pR^2 - p + 2(p-\alpha)R^2\cos^2\lambda - \beta)\sin\frac{\pi}{2}\gamma - (p-\alpha)R^2\sin 2\lambda\cos\frac{\pi}{2}\gamma\right] \\ &- r[|\beta|R + 2(p-\alpha)R^2\cos\lambda + 2(\delta+1)R^2] + pR^2\sin\frac{\pi}{2}\gamma\end{aligned}$$

Since $T(0) > 0$ and $T(1) < 1$, there exists a real root of $T(r) = 0$ in $(0,1)$. Let $R(\gamma)$ be the smallest positive root of $T(r)=0$ in $(0,1)$. Then F is p-valent strongly starlike of order γ in $|z| < R(\gamma)$.

Remark. *For $R = 1$ and $\gamma = 1$, the about theorem reduces to a result of J. Patel.*

Theorem 2. *Suppose $F(z)$ is given by (3.1). If $f(z) \in A_p$ satifies (3.2) for some $g(z) \in S_p^\lambda(\alpha)$, then $F(z)$ is p-valent strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is the samallest positive root of the equation*

$$\begin{aligned}
&r^4\left[(p+\beta)\sin\frac{\pi}{2}\gamma + 2(p-\alpha)\cos\lambda\sin(\lambda-\frac{\pi}{2}\gamma)\right]\\
&+r^3[|\beta|R + 2(p-\alpha)\cos\lambda + 2\delta]\\
&-r^2\left[(p(1+R^2)+\beta)\sin\frac{\pi}{2}\gamma + 2(p-\alpha)R^2\cos\lambda\sin(\lambda-\frac{\pi}{2}\gamma)\right]\\
&-r[|\beta|R + 2(p-\alpha)R^2\cos\lambda + 2\delta R^2]\\
&+pR^2\sin\frac{\pi}{2}\gamma.
\end{aligned} \tag{3.9}$$

Proof. If $f(z) \in A_p$ satisfies (3.2) for some $g(z) \in S_p^\lambda(\alpha)$, then

$$\frac{zF'(z)}{F(z)} = \delta\cdot\frac{zp'(z)}{p(z)} + \frac{zg'(z)}{g(z)} + \frac{\beta}{n}\sum_{k=1}^{n}\frac{z}{z-z_k}. \tag{3.10}$$

Using Lemma 4, we get

$$\left|\frac{zg'(z)}{g(z)} - \left\{p + \frac{2(p-\alpha)e^{i\lambda}r^2\cos\lambda}{1-r^2}\right\}\right| \le \frac{2(p-\alpha)r\cos\lambda}{1-r^2}. \tag{3.11}$$

By (3.10) and (3.11) and Lemma 1, 3, we have

$$\begin{aligned}
&\left|\frac{zF'(z)}{F(z)} - \left\{p + \frac{2(p-\alpha)e^{i\lambda}r^2\cos\lambda}{1-r^2} - \frac{\beta r^2}{R^2-r^2}\right\}\right|\\
\le&\frac{2\{(p-\alpha)r\cos\lambda + r\delta\}}{1-r^2} + \frac{|\beta|Rr}{R^2-r^2}.
\end{aligned}$$

The remaining parts of the proof can be proved by similar method given in the Theorem 1.

With $\lambda = 0$, $\beta = 0$, $\delta = 1$, $R = 1$ and $\gamma = 1$, Theorem 2 gives

Corollary 1. *Suppose $f(z)$ is in A_p. If $Re\left(\frac{f(z)}{g(z)}\right) > 0$ for $z \in D$ and $g(z) \in S_p^*(\alpha)$, then $f(z)$ is p-valent starlike for*

$$|z| < \frac{p}{(p+1-\alpha)+\sqrt{\alpha^2-2\alpha+2p+1}}.$$

Theorem 3. *Suppose $F(z)$ is given by (3.1). If $f(z) \in A_p$ satisfies*

$$\left|\left(\frac{f(z)}{g(z)}\right)^{\frac{1}{\delta}} - 1\right| < 1, \quad 0 < \delta \leq 1, \quad p \sin \frac{\pi}{2}\gamma > \delta \tag{3.12}$$

and

$$Re\left(\frac{g(z)}{h(z)}\right) > 0, \quad z \in D$$

for some $g(z) \in A_p$ and $h(z) \in S_p^\lambda(\alpha)$, then $F(z)$ is p-valent strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

$$\begin{aligned}
&r^4\left[(p+\beta)\sin\frac{\pi}{2}\gamma + 2(p-\alpha)\cos\lambda\sin(\lambda - \frac{\pi}{2}\gamma)\right] \\
&+r^3[|\beta|R + 2(p-\alpha)\cos\lambda + 2 + \delta] \\
&-r^2\left[(p(1+R^2)+\beta)\sin\frac{\pi}{2}\gamma + 2(p-\alpha)R^2\cos\lambda\sin(\lambda-\frac{\pi}{2}\gamma) + \delta\right] \\
&-r[|\beta|R + 2(p-\alpha)R^2\cos\lambda + 2(\delta+1)R^2] + pR^2\sin\frac{\pi}{2}\gamma - \delta R^2.
\end{aligned} \tag{3.13}$$

Proof. We choose a suitable branch of $\left(\frac{f(z)}{g(z)}\right)^{\frac{1}{\delta}}$ so that $\left(\frac{f(z)}{g(z)}\right)^{\frac{1}{\delta}}$ is analytic in D and takes the value 1 at $z = 0$. From (3.12), we deduce taht

$$f(z) = g(z) \cdot (1 + w(z))^\delta, \quad \text{where} \quad w(z) \in \Omega.$$

So that

$$F(z) = p(z) \cdot h(z) \cdot (1 + z\phi(z))^\delta [Q(z)]^{\frac{\beta}{n}}$$

where $\phi(z)$ is analytic in D and satisfies $|\phi(z)| \leq 1$ and $p \in P$ for $z \in D$.

We have

$$\frac{zF'(z)}{F(z)} = \frac{zh'(z)}{h(z)} + \frac{zp'(z)}{p(z)} + \delta\left(\frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)}\right) + +\frac{\beta}{n}\sum_{k=1}^{n}\frac{z}{z - z_k}. \tag{3.14}$$

Using Lemma 4 and (3.14), we have

$$\begin{aligned}
&\left|\frac{zF'(z)}{F(z)} - \left\{p + \frac{2(p-\alpha)e^{i\lambda}r^2\cos\lambda}{1-r^2}\right\}\right| \\
&\leq \frac{2\{(p-\alpha)r\cos\lambda + r\} + \delta(1+r)}{1-r^2} + \frac{|\beta|Rr}{R^2 - r^2}
\end{aligned} \tag{3.15}$$

So, using Lemma 5 and (3.15), the result can be proved by similar method given in the Theorem 1.

Theorem 4. *Suppose $F(z)$ is given by (3.1). If $f(z) \in A_p$ satisfies (3.12) for some $g(z) \in S_p^{\lambda}(\alpha)$, then $F(z)$ is p-valent strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is smallest positive root of the equation*

$$\begin{aligned} & r^4\left[(p+\beta)\sin\frac{\pi}{2}\gamma + 2(p-\alpha)\cos\lambda\sin(\lambda - \frac{\pi}{2}\gamma)\right] \\ & +r^3[|\beta|R + 2(p-\alpha)\cos\lambda + \delta] \\ & -r^2\left[(p(1+R^2)+\beta)\sin\frac{\pi}{2}\gamma + 2(p-\alpha)R^2\cos\lambda\sin(\lambda-\frac{\pi}{2}\gamma) + \delta\right] \\ & -r[|\beta|R + 2(p-\alpha)R^2\cos\lambda + \delta R^2] \\ & +pR^2\sin\frac{\pi}{2}\gamma - \delta R^2. \end{aligned} \tag{3.16}$$

Proof. We choose a suitable of $(f(z)/g(z))^{\frac{1}{\delta}}$ so that $(f(z)/g(z))^{\frac{1}{\delta}}$ is analytic in D and takes the value 1 at $z = 0$. Since $f(z) \in A_p$ (3.12) for some $g(z) \in S_p^{\lambda}(\alpha)$, we have

$$F(z) = g(z)(1 + z\phi(z))[Q(z)]^{\frac{\beta}{n}}$$

where $\phi(z)$ is analytic in D and satisfies the condition $|\phi(z)| \leq 1$ for $z \in D$. Thus, we have

$$\frac{zF'(z)}{F(z)} = \frac{zg'(z)}{g(z)} + \delta\left(\frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)}\right) + +\frac{\beta}{n}\sum_{k=1}^{n}\frac{z}{z - z_k}. \tag{3.17}$$

Using Lemma 4 and (3.17), we get

$$\begin{aligned} & \left|\frac{zF'(z)}{F(z)} - \left\{p + \frac{2(p-\alpha)e^{i\lambda}r^2\cos\lambda}{1-r^2}\right\}\right| \\ \leq & \frac{2(p-\alpha)r\cos\lambda + \delta(1+r)}{1-r^2} + \frac{|\beta|Rr}{R^2-r^2} \end{aligned} \tag{3.18}$$

Using Lemma 5 and (3.18) and similar method in the Theorem 1, we get the Theorem4.

Remark. *Some of the results of J. Patel can be obtained form the Theorem 4 by taking $R = 1$, $\gamma = 1$.*

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